NOTES ON HARMONIC ANALYSIS
PART I: THE FOURIER TRANSFORM

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Abstract. Fourier Transforms is a first in a series of monographs we present on harmonic analysis. Harmonic analysis is one of the most fascinating areas of research in mathematics. Its centrality in the development of many areas of mathematics such as partial differential equations and integration theory and its many and diverse applications in sciences and engineering fields makes it an attractive field of study and research.

The purpose of these notes is to introduce the basic ideas and theorems of the subject to students of mathematics, physics or engineering sciences. Our goal is to illustrate the topics with utmost clarity and accuracy, readily understandable by the students or interested readers. Rather than providing just the outlines or sketches of the proofs, we have actually provided the complete proofs of all theorems. This will illuminate the necessary steps taken and the machinery used to complete each proof.

The prerequisite for understanding the topics presented is the knowledge of Lebesgue measure and integral. This will provide ample mathematical background for an advanced undergraduate or a graduate student in mathematics.

1. Fourier Transforms for $L^1(\mathbb{R})$

Definition 1.1. *For $f \in L^1(\mathbb{R})$, the Fourier transform $\hat{f}$ of f is defined as*

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-ixy} dx \tag{1.1}$$

for all real $y \in \mathbb{R}$.

It is easy to see that Fourier transform is a lineaer operator, i.e., $(f+g)\hat{}(y) = \hat{f}(y) + \hat{g}(y)$ and $(kf)\hat{}(y) = k\hat{f}(y)$. Also using simple integration techniques it can easily be shown that $\hat{f}(y+t) = e^{ity}\hat{f}(y)$ and $\hat{f}(ky) = \frac{1}{k}\hat{f}(\frac{y}{k})$.

Theorem 1.1. *If $f \in L^1(\mathbb{R})$, then $\hat{f}(y)$ is uniformly continuous and bounded in $\mathbb{R}$.*

Proof: Clearly, $|\hat{f}(y)| \leq ||f||_1$ for all y. Moreover,

$$\begin{aligned} |\hat{f}(y+h) - \hat{f}(y)| &= |\int_{-\infty}^{\infty} f(x)(e^{-ix(y+h)} - e^{-ixy})dx| \\ &\leq \int_{-\infty}^{\infty} |f(x)||e^{-ixh} - 1|dx. \end{aligned}$$

The integrand on the right side converges to 0 as $h \to 0$ and is dominated by $2|f(x)| \in L^1(\mathbb{R})$. So, by Lebesgue's dominated convergence theorem, $\hat{f}$ is uniformly continuous. $\square$

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Theorem 1.2 (Riemann-Lebesgue Lemma)**.** *If $f \in L^1(\mathbb{R})$, then $\hat{f}(y) \to 0$ as $y \to \pm\infty$.*

Proof: First suppose that f is a characteristic function of an interval $[a, b]$. Its Fourier transform is

$$\int_a^b e^{-ixy} dx = \frac{e^{-iay} - e^{-iby}}{iy}, \quad y \neq 0,$$

which tends to zero. Therefore, a linear combination of characteristic functions of intervals, i.e., a step function, satisfies the Riemann-Lebesgue lemma. Such functions are also dense in $L^1(\mathbb{R})$. Now let $f \in L^1(\mathbb{R})$ and let $f_n \in L^1(\mathbb{R})$ be a sequence of step functions such that $f_n \to f$ in $L^1(\mathbb{R})$. Then

$$|\hat{f}_n(y) - \hat{f}(y)| = |(f_n - f)\hat{}(y)| \leq ||f_n - f||_1 \to 0.$$

Note that the limit is uniform in $y \in \mathbb{R}$. Since

$$|\hat{f}(y)| \leq |\hat{f}_n(y) - \hat{f}(y)| + |\hat{f}_n(y)|,$$

we can choose n large enough so that the first term on the right is small and then for that fixed n, we let $|y|$ large enough so that the second term is also small. This completes the proof. $\square$

Theorem 1.3. *Suppose that $f(x)(1 + |x|)$ is integrable. Then,*

$$(\hat{f})'(y) = (-ixf(x))\hat{}(y). \tag{1.2}$$

Proof: Note that, by assumption, both f and $xf(x)$ are integrable. We write

$$\begin{aligned}(\hat{f})'(y) &= \lim_{h\to 0} \int_{-\infty}^{\infty} f(x) \frac{e^{-ix(y+h)} - e^{-ixy}}{h} dx \\ &= \lim_{h\to 0} \int_{-\infty}^{\infty} f(x) e^{-ixy} \frac{e^{-ixh} - 1}{h} dx.\end{aligned}$$

Note that the integrand converges to $f(x)e^{-ixy}(-ix)$ pointwise as $h \to 0$ and $|f(x)e^{-ixy}\frac{e^{-ixh}-1}{h}| \leq |xf(x)|$ for all small $|h|$. [1] Hence, by Lebesgue's dominated convergence theorem ,

$$(\hat{f})'(y) = \lim_{h\to 0} \int_{-\infty}^{\infty} f(x) e^{-ixy} \frac{e^{-ixh} - 1}{h} dx = \int_{-\infty}^{\infty} (-ixf(x)) e^{-ixy} dx. \quad \square$$

Theorem 1.4. *If f is continuously differentiable with compact support, then*

$$(f')\hat{}(y) = iy\hat{f}(y). \tag{1.3}$$

Proof: Integration by parts. $\square$

Definition 1.2. *The convolution of f and g is defined as*

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - t) g(t) dt, \tag{1.4}$$

[1] Estimating the remainder (both Lagrange form and integral form) of Taylor's series for e^{ix} we obtain the estimation

$$|e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!}| \leq \min(\frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!}).$$

Note that the first estimate is better for small $|x|$, while the second is better for large $|x|$. Choosing $n = 0$ and considering small $|h|$ we get the inequality in context.

whenever the integral exists.

In the following, $C(\mathbb{R})$ denotes the space of all continuous functions on $\mathbb{R}$ with $||f||_C = sup_{x\in\mathbb{R}}|f(x)| < \infty$ and $C_0(\mathbb{R})$ the space of all continuous functions on $\mathbb{R}$ that vanishes at infinity, i.e., for any $\epsilon > 0$, there is a compact $F \subset \mathbb{R}$ such that $|f(x)| < \epsilon$ for $x \notin F$. Then by F. Riesz' theorem, $(C_0(\mathbb{R}))^* = M(\mathbb{R})$, where $M(\mathbb{R})$ is the space of complex regular Borel measures on R.[2] Since $C_0(\mathbb{R})$ is separable (continuous functions with compact support are dense in $C_0(\mathbb{R})$), every bounded subset of $M(\mathbb{R})$ is weak* sequentially compact. Note that $L^1(\mathbb{R})$ is contained in $M(\mathbb{R})$, if we identify $f \in L^1(\mathbb{R})$ with the measure $f(x)dx$.

Theorem 1.5. *Let $f \in L^p(\mathbb{R})$, $1 \le p \le \infty$, and $g \in L^q, \frac{1}{p} + \frac{1}{q} = 1$. Then $(f * g)(x)$ exists everywhere, belongs to $C(\mathbb{R})$, and $||f * g||_C \le ||f||_p||g||_q$. Moreover, if $1 < p < \infty$, or if $p = 1$ and $g \in C_0(\mathbb{R})$, then $f * g \in C_0(\mathbb{R})$, i.e., $f * g \in C(\mathbb{R})$ and $\lim_{|x|\to\infty} |(f * g)(x)| = 0$.*

Proof: Let $1 \le p < \infty$. By Hölder's inequality, $|(f * g)(x)| \le ||f||_p||g||_q$ and so $(f * g)(x)$ exists for every $x \in \mathbb{R}$. Furthermore,

$$|(f * g)(x + h) - (f * g)(x)| \le ||f(\cdot + h) - f(\cdot)||_p||g||_q,$$

and therefore $f * g \in C(\mathbb{R})$ by the continuity of f in mean. If $p = \infty$, the roles of f and g can be interchanged.

Now let $1 < p < \infty$ (obviously $1 < q < \infty$ also). Given $\epsilon > 0$, there is a finite interval $[-a, a]$ such that

$$\int_{|t|\ge a} |f(t)|^p dt \le \epsilon^p \text{ and } \int_{|t|\ge a} |g(t)|^q dt \le \epsilon^q.$$

If $x \in \mathbb{R}$ is such that $|x| > 2a$, then $[x - a, x + a]$ is contained in $\{t \in \mathbb{R} : |t| > a\}$, and hence

$$\begin{aligned} |(f * g)(x)| &\le (\int_{-a}^{a} + \int_{|t|\ge a})|f(x - t)g(t)|dt \\ &\le (\int_{-a}^{a} |f(x - t)|^p dt)^{1/p}||g||_q + ||f||_p(\int_{|t|\ge a} |g(t)|^q dt)^{1/q} \\ &\le (\int_{x-a}^{x+a} |f(t)|^p dt)^{1/p}||g||_q + ||f||_p\epsilon \\ &\le (\int_{|t|>a} |f(t)|^p dt)^{1/p}||g||_q + ||f||_p\epsilon \le \epsilon(||g||_q + ||f||_p). \end{aligned}$$

Thus, $(f * g)(x)$ tends to 0 as $|x| \to \infty$, giving $f * g \in C_0(\mathbb{R})$. The same method of proof applies for the case $p = 1$, $g \in C_0(\mathbb{R})$. □

Theorem 1.6. *If $f, g \in L^1(\mathbb{R})$, then $(f * g)(x)$ exists a.e. and $||f * g||_1 \le ||f||_1||g||_1$. Moreover,*

$$(f * g)\hat{}(y) = \hat{f}(y) \cdot \hat{g}(y). \tag{1.5}$$

Proof: Note that the integral $\int |f(x - t)g(t)|dx$ exists for a.e. t and

$$\int |f(x - t)g(t)|dx = |g(t)| \cdot ||f||_1.$$

[2] Every complex measure is bounded, see Hewitt and Stromberg [3] .

Also note that the expression on the right belongs to $L^1(\mathbb{R})$. Hence, the integral

$$\int(\int |f(x-t)g(t)|dx)dt = ||f||_1||g||_1$$

exists as a finite number. Therefore, by Fubini's theorem the integral

$$\int(\int |f(x-t)g(t)|dt)dx$$

exists and is equal to $||f||_1||g||_1$. This implies that $\int |f(x-t)g(t)|dt$ exists a.e. and belongs to L^1.

To prove $(f*g)\hat{}(y) = f\hat{}(y)\cdot g\hat{}(y)$, we observe that

$$\begin{aligned}(f*g)\hat{}(y) &= \int(\int f(x-t)g(t)dt)e^{-ixy}dx \\ &= \int g(t)e^{-ity}(\int f(x-t)e^{-iy(x-t)}dx)dt = \hat{f}(y)\hat{g}(y).\end{aligned}$$

The change in the order of integration is justified by Fubini's theorem. □

It is easy to see that convolution obeys the commutative and distributive laws of algebra in $L^1(\mathbb{R})$, i.e., $f*g = g*f$ and $f*(g+h) = f*g + f*h$. The natural question is whether there is a multiplicative identity, i.e., given $f \in L^1(\mathbb{R})$, is there $e \in L^1(\mathbb{R})$ such that $f*e = f$? The answer is, in general, no since convolution exhibits continuity property and cannot be equal to a discontinuous f. However, we may seek a sequence of functions e_n, called approximate identity, with the property that $e_n * f \to f$.

Definition 1.3. *An approximate identity e_n on $\mathbb{R}$ is a sequence of functions e_n such that $e_n \geq 0$, $\int e_n(x)dx = 1$, and for each $\delta > 0$,*

$$\lim_{n\to\infty}\int_{|x|>\delta} e_n(x)dx = 0.$$

Theorem 1.7. *If $f \in C_0(\mathbb{R})$, then $e_n * f \to f$ uniformly. If $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, then $e_n * f \to f$ in $L^p(\mathbb{R})$. If $f \in L^\infty(\mathbb{R})$, then $e_n * f \to f$ in the weak* topology of $L^\infty(\mathbb{R})$ as a dual of $L^1(\mathbb{R})$, that is, $\int(e_n * f)(x)g(x)dx \to \int f(x)g(x)dx$ for all $g \in L^1(\mathbb{R})$.*

Proof: Note that if $f \in C_0(\mathbb{R})$, then f is uniformly continuous on $\mathbb{R}$ and for any given $\epsilon > 0$, there is a $\delta > 0$ such that for any t with $|t| < \delta$, $|f(x-t) - f(x)| < \epsilon$ for all $x \in \mathbb{R}$. Hence,

$$\begin{aligned} & |(e_n * f)(x) - f(x)| \leq \int_R |f(x-t) - f(x)|e_n(t)dt \\ = & \int_{|t|<\delta} |f(x-t) - f(x)|e_n(t)dt + \int_{|t|\geq\delta} |f(x-t) - f(x)|e_n(t)dt \\ \leq & \ \epsilon + 2M\int_{|t|>\delta} e_n(t)dt,\end{aligned}$$

where $M = \sup_{x\in\mathbb{R}} |f(x)|$. Since $\lim_{n\to\infty} \int_{|x|>\delta} e_n(x)dx = 0,\ e_n * f \to f$ uniformly. In the case of $f \in L^\infty(\mathbb{R})$, the proof is similar.

If $f \in L^p(\mathbb{R}), 1 \le p < \infty$, then

$$\begin{aligned}\int_{\mathbb{R}} |(e_n * f)(x) - f(x)|^p dx &\le \int_{\mathbb{R}} |\int_{\mathbb{R}} (f(x-t) - f(x))e_n(t)dt|^p dx \\ &\le \int_{\mathbb{R}} (\int_{\mathbb{R}} |f(x-t) - f(x)|^p\, |e_n(t)|dx)dt \\ &= \int_{\mathbb{R}} ||f(\cdot - t) - f(\cdot)||_p^p\, |e_n(t)|dt.\end{aligned}$$

Given any $\epsilon > 0$, there is a $\delta > 0$ such that $||f(\cdot - t) - f(\cdot)||_p < \epsilon$ whenever $|t| < \delta$. Hence,

$$\begin{aligned}&\int_R ||f(\cdot - t) - f(\cdot)||_p^p\, |e_n(t)|dt \\ =\ &\int_{|t|<\delta} ||f(\cdot - t) - f(\cdot)||_p^p\, e_n(t)dt + \int_{|t|\ge\delta} ||f(\cdot - t) - f(\cdot)||_p^p\, e_n(t)dt \\ \le\ &\epsilon^p + 2||f||_p^p \int_{|t|>\delta} e_n(t)dt.\end{aligned}$$

Since $\lim_{n\to\infty} \int_{|x|>\delta} e_n(x)dx = 0,$ the result follows. □

Theorem 1.8. *If f has compact support and a continuous derivative, and $g \in L^1(\mathbb{R})$, then $f * g \in L^1(\mathbb{R})$ has a continuous derivative.*

Proof: First, we prove

$$\frac{d}{dx}(\int f(x-t)g(t)dt) = \int f'(x-t)g(t)dt,$$

which is showing that

$$\lim_{h\to 0} \int_{-\infty}^{\infty} (\frac{f(x+h-t) - f(x-t)}{h})g(t)dt = \int f'(x-t)g(t)dt.$$

Note that the integrand on the left converges to $f'(x-t)g(t)$ pointwise (in t) as $h \to 0$. Moreover, $\frac{f(x+h-t)-f(x-t)}{h} = f'(c)$, where c is between $x+h-t$ and $x-t$. If f has compact support S, then so does f'. Therefore, $|\frac{f(x+h-t)-f(x-t)}{h}| = |f'(c)| \le sup_{c\in S}|f'(c)| \le M$ with some $M > 0$ for all $t \in (-\infty, \infty)$. Now the desired limit follows from Lebesgue's dominated convergence theorem.

To prove that $\int f'(x-t)g(t)dt$ is continuous, we note that

$$\begin{aligned}&|\int f'(x+h-t)g(t)dt - \int f'(x-t)g(t)| \\ =\ &|\int f'(t)(g(x+h-t) - g(x-t))dt| \\ \le\ &||f'||_c||g(\cdot + h) - g(\cdot)||_1.\end{aligned}$$

Then the (uniform) continuity of $\displaystyle\int f'(x-t)g(t)dt$ follows from the continuity of g in mean.

□

The following corollary follows immediately from Theorems 1.7 and 1.8.

Corollary 1.1. *Let $e_n(x)$ be an approximate identity with compact support and continuous derivative. Then for any $f \in L^1(\mathbb{R})$, $e_n * f$ provides a continuously differentiable approximation to f in $L^1(\mathbb{R})$.*

Proof: An obvious result from Theorem 1.8. □

Theorem 1.9. *Let $\phi(x) \geq 0$ be a function defined on $\mathbb{R}$ such that ϕ has compact support and continuous derivative, and $\int \phi(x)dx = 1$. Then $e_n(x) = n\phi(nx)$ is an approximate identity with compact support and continuous derivative.*

Proof: We only need to show that for each $\epsilon > 0$, $\displaystyle\int_{|t|\geq\epsilon} e_n(t)dt = 0$. In fact,

$$\int_{|t|\geq\epsilon} n\phi(nt)dt = \int_{|u|\geq n\epsilon} \phi(u)du \to 0,$$

as $n \to \infty$. □

Theorem 1.10. *If f and $\dfrac{f(x)}{x}$* [3] *are both integrable, then*

$$\lim_{A,B\to\infty} \int_{-B}^{A} \hat{f}(y)dy = 0.$$

Proof: Observe that

$$\int_{-B}^{A} \hat{f}(y)dy = \int_{-B}^{A}\int_{-\infty}^{\infty} f(x)e^{-ixy}dxdy.$$

The integrand on the right side is integrable over the product space, so, by Fubini's theorem,

$$\begin{aligned}\int_{-B}^{A}\int_{-\infty}^{\infty} f(x)e^{-ixy}dxdy &= \int_{-\infty}^{\infty}\int_{-B}^{A} f(x)e^{-ixy}dydx \\ &= \int_{-\infty}^{\infty} f(x)\frac{e^{iBx}-e^{-iAx}}{ix}dx.\end{aligned}$$

The last integral tends 0 (as $A, B \to \infty$), by Riemann-Lebesgue lemma. □

To derive the following inversion theorem, we need a simple fact, which can be verified by a straightforward calculation: If $g(x) = e^{-|x|}$, then $\hat{g}(y) = \dfrac{2}{1+y^2}$ and $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\hat{g}(y)dy = 1$.

Corollary 1.2. *If f is integrable in $\mathbb{R}$ and satisfies a Lipschitz condition at t, then*

$$f(t) = \lim_{A,B\to\infty}\frac{1}{2\pi}\int_{-B}^{A}\hat{f}(y)e^{ity}dy.$$

[3] The assumption that $\dfrac{f(x)}{x} \in L^1(\mathbb{R})$ simply emphasizes that f behaves like a *positive* power of x at $x = 0$. For example, If $f \in Lip(\alpha)$ for $0 < \alpha \leq 1$ at $x = 0$, and $f(0) = 0$, then $\dfrac{f(x)}{x} \in L^1(\mathbb{R})$.

That is, f is the inverse Fourier transform of $\hat{f}$.

Proof: If $t \neq 0$, let $h(x) = f(x+t)$. If $f(x)$ satisfies $|f(x) - f(t)| \leq K|x-t|^\alpha$ for x near t then for x near 0, $h(x)$ satisfies the Lipschitz condition at $t = 0$: $|h(x) - h(0)| = |f(x+t) - f(t)| \leq K|x|^\alpha$. Therefore, if we can show the corollary for $t = 0$, then for $t \neq 0$,

$$f(t) = h(0) = \lim_{A,B\to\infty} \frac{1}{2\pi} \int_{-B}^{A} \hat{h}(y)dy = \lim_{A,B\to\infty} \frac{1}{2\pi} \int_{-B}^{A} \hat{f}(y)e^{ity}dy.$$

We may now assume that $t = 0$. Since f satisfies the Lipschitz condition at 0, if $f(0) = 0$, it follows that $\frac{f(x)}{x} \in L^1(\mathbb{R})$. Then by Theorem 1.10, $\lim_{A,B\to\infty} \int_{-B}^{A} \hat{f}(y)dy = 0$, which shows that

$$f(t) = \lim_{A,B\to\infty} \frac{1}{2\pi} \int_{-B}^{A} \hat{f}(y)e^{ity}dy$$

holds as $t = 0$ and $f(0) = 0$.

If $f(0) \neq 0$, we reduce it to the case $f(0) = 0$. Let $\phi(x) = f(x) - f(0)g(x)$, where $g(x) = e^{-|x|}$. Then $\phi(0) = 0$ and $\phi(x)$ satisfies the Lipschitz condition at $t = 0$. Therefore,

$$\frac{1}{2\pi} \int_{-B}^{A} \hat{\phi}(y)dy \to 0,$$

as $A, B \to \infty$. That is,

$$\frac{1}{2\pi} \int_{-B}^{A} \hat{f}(y)dy - \frac{1}{2\pi} \int_{-B}^{A} f(0)\hat{g}(y)dy \to 0.$$

It follows that

$$\frac{1}{2\pi} \int_{-B}^{A} \hat{f}(y)dy \to f(0). \quad \square$$

Remark: For $f \in L^1(\mathbb{R})$, $\hat{f}$ need not be in $L^1(\mathbb{R})$. Therefore, the above integral $\int_{-B}^{A} \hat{f}(y)e^{ity}dy$ has to be understood as the limit of the integral from $-B$ to A as $A, B \to \infty$. Note that a function $f \geq 0$ is integrable on $\mathbb{R}$ if $\lim_{A,B\to\infty} \int_{-B}^{A} f(x)dx$ exists. Therefore, a function f is integrable on $\mathbb{R}$ if both f^+ and f^- are integrable on $\mathbb{R}$. By this definition, f and $|f|$ are either both integrable or not integrable. Hence, it may happen that $\hat{f} \notin L^1(\mathbb{R})$, yet the above limit exists. Let $m(x) = \chi_{[-a,a]}(x)$. Clearly, $\hat{m}(y) = \frac{2\sin ay}{y} \notin L^1(\mathbb{R})$, but

$$m(x) = \lim_{A,B\to\infty} \frac{1}{2\pi} \int_{-B}^{A} \hat{m}(y)e^{ixy}dy, \quad x \neq \pm 1.$$

With improper Riemann integral in mind, we may say f equals the inverse Fourier transform of $\hat{f}$ at each Lipschitz point.

Theorem 1.11.

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1+t^2} e^{ixt}dt = e^{-|x|}. \tag{1.6}$$

Proof: Assuming that $x > 0$ and integrating

$$I = \frac{1}{2\pi}\int_{\Gamma_R}\frac{2}{1+z^2}e^{ixz}dz,$$

where Γ_R consists of the upper semicircle C_R and the line segment $[-R, R]$ on the x-axis, we see that

$$I = Res_{z=i}\frac{2}{1+z^2}e^{ixz} = e^{-x}$$

and that the integral along $[-R, R]$ gives

$$I = \frac{1}{2\pi}\int_{\Gamma_R}\frac{2}{1+t^2}e^{ixt}dt,$$

while, if $x > 0$, then

$$\frac{1}{2\pi}\int_{C_R}\frac{2}{1+z^2}e^{ixz}dz \to 0$$

as $R \to \infty$.[4] Thus, if $x > 0$, $\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2}{1+t^2}e^{ixt}dt = e^{-x}$. Similarly, if $x < 0$ then $\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2}{1+t^2}e^{ixt}dt = e^{x}$. Hence,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2}{1+t^2}e^{ixt}dt = e^{-|x|}. \quad \square$$

In the following, we will calculate the Fourier transform of a Gaussian function which will be useful in proving the inversion theorem. The theorem below simply states that Fourier transform of a Gaussian function is a Gaussian.

Theorem 1.12.

$$(e^{-x^2})\hat{}(y) = \sqrt{\pi}e^{-y^2/4}. \tag{1.7}$$

Proof: First, let u be real. We have

$$\int_{-\infty}^{\infty} e^{-x^2+2xu}dx = e^{u^2}\int_{-\infty}^{\infty} e^{-(x-u)^2}dx = e^{u^2}\int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}e^{u^2}.$$

Clearly, the function defined by

$$\int_{-\infty}^{\infty} e^{-x^2+2xz}dx$$

is an entire function[5], and by above calculation, it coincides with the entire function $\sqrt{\pi}e^{z^2}$ along the x-axis. Therefore, for all z,

$$\int_{-\infty}^{\infty} e^{-x^2+2xz}dx = \sqrt{\pi}e^{z^2}.$$

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[4](Jordan's Lemma) Suppose that f is an analytic function in the upper half plane except at a finite number of singularities and $|f(z)| \to 0$ as $|z| \to \infty$ for $0 \le Arg(z) \le \pi$. Then, if $x > 0$, $\int_{C_R} e^{ixz}f(z)dz \to 0$ as $R \to \infty$.

[5]See the theorem in complex analysis. Suppose that $f(z, w)$ is a continuous function of $z \in D$ and $w \in C$, where D is a region and C is a contour that is a piecewise smooth curve $w(t) = u(t) + iv(t)$, $t_0 \le t \le t_1$, with continuous u' and v'. Suppose that for each $w \in C$, $f(z, w)$ is an analytic function in $z \in D$. Then $F(z) = \int_C f(z, w)dw$ is analytic in D and $F'(z)$ can be found by differentiating under the integral sign.

If C is a contour going to infinity such that any bounded part of it is regular (no sharp corner) and if the above conditions are satisfied on any bounded part of C, and if $\int_C f(z, w)dw$ converges uniformly in $z \in D$, then the above results hold.

In particular, let $z = \frac{-iy}{2}$. Then we have

$$\int_{-\infty}^{\infty} e^{-x^2-ixy}dx = \sqrt{\pi}e^{-y^2/4}. \quad \square$$

Theorem 1.13 (Inversion Theorem). *Let $f \in L^1(\mathbb{R})$, and $\hat{f} \in L^1(\mathbb{R})$, then*

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(y)e^{ixy}dy \tag{1.8}$$

for almost all real $x \in \mathbb{R}$. The integral is commonly known as the inverse Fourier transform.

Proof: Consider the Gauss-Weierstrass Kernel, $W(x,\alpha) = \frac{1}{\sqrt{\pi\alpha}}e^{-\frac{x^2}{\alpha}}$. A straightforward calculation shows that $W(\cdot,\alpha)\hat{}(t) = e^{-\frac{\alpha t^2}{4}}$. By integrating $\hat{f}$ against $W\hat{}$, and then applying Fubini's theorem and the fact that $W(x,\alpha)$ is an approximate identity, we get

$$\begin{aligned}
&\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi x}W(\cdot,\alpha)\hat{}(\xi)d\xi \\
= &\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} f(t)e^{-i\xi t}dt)e^{i\xi x}e^{-\frac{\alpha\xi^2}{4}}d\xi \\
= &\int_{-\infty}^{\infty} f(t)(\int_{-\infty}^{\infty} e^{-\frac{\alpha\xi^2}{4}}e^{-i\xi(t-x)}d\xi)dt \\
= &\int_{-\infty}^{\infty} f(t)2\pi W(t-x,\alpha)dt \\
= &\ 2\pi\int_{-\infty}^{\infty} f(x-t)W(t,\alpha)dt \to 2\pi f(x) \ \ a.e. \ \text{ as } \ \alpha \to 0^+
\end{aligned}$$

On the other hand, by Lebesgue's dominated convergence theorem,

$$\lim_{\alpha\to 0^+}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi x}W(\cdot,\alpha)\hat{}(\xi)d\xi = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi x}d\xi.$$

The theorem follows. $\square$

As an application of the inversion theorem, we now prove that the Fourier transform of a product is the convolution of the Fourier transforms.

Theorem 1.14. *Assume that $f, g \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$ (or $\hat{g} \in L^1(\mathbb{R})$). Then,*

$$(fg)\hat{}(x) = \frac{1}{2\pi}(\hat{f} * \hat{g})(x). \tag{1.9}$$

Proof: By the inversion theorem, f is bounded and so, $fg \in L^1(\mathbb{R})$. Hence,

$$\begin{aligned}
(fg)\hat{}(x) &= \int_{-\infty}^{\infty} f(y)g(y)e^{-ixy}dy \\
&= \int_{-\infty}^{\infty} g(y)e^{-ixy}(\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(t)e^{iyt}dt)dy \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(t)(\int_{-\infty}^{\infty} g(y)e^{-ixy}e^{iyt}dy)dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(t)\hat{g}(x-t)dt \\
&= \frac{1}{2\pi}(\hat{f} * \hat{g})(x).
\end{aligned}$$

The change in the order of integration is justified by Fubini's theorem, since due to boundedness of $\hat{f}$, $\hat{f}g \in L^1(\mathbb{R})$. □

We now investigate the question of uniqueness of Fourier transform, i.e, $\hat{f} = \hat{g}$ implies $f = g$. To show this, since Fourier transform is a linear operator, it suffices to show that $\hat{f} = 0$ implies $f = 0$ a.e.

Theorem 1.15 (Uniqueness Theorem)**.** *If $f \in L^1(\mathbb{R})$ and $\hat{f} = 0$ everywhere ($\hat{f}$ is always continuous), then $f = 0$ a.e.*

Proof: Let $e_n(x)$ be an approximate identity with compact support and continuous derivative. By Theorem 1.6, $(e_n * f)\hat{} = \hat{e_n}\hat{f} = 0$ everywhere. Since by Theorem 1.8, $e_n * f$ is continuous and differentiable, by the inversion theorem, $e_n * f = 0$ everywhere. But by Theorem 1.7, $e_n * f \to f$ in $L^1(\mathbb{R})$; so it follows that $f = 0$ a.e. □

Definition 1.4. *For $\mu \in M(\mathbb{R})$ (bounded Borel measure on $\mathbb{R}$, i.e., $|\mu|(\mathbb{R}) < \infty$), define the Fourier-Stieltjes transform $\mu\hat{}(y)$ as*

$$\mu\hat{}(y) = \int_{-\infty}^{\infty} e^{ixy}d\mu(x).$$

Clearly, the Fourier-Stieltjes transform defines a bounded linear transform from $M(\mathbb{R})$ to $\mathbb{C}$.

Theorem 1.16 (Uniqueness Theorem)**.** *If $\hat{\mu}(y) = 0$ for a.e. y, then $\mu = 0$.*

Proof: Since $(C_0(\mathbb{R}))^* = M(\mathbb{R})$, to prove $\mu = 0$ we need only to show that for all $h \in C_0(\mathbb{R})$, $\displaystyle\int h(t)d\mu(t) = 0$. This is equivalent to showing that for all $h \in C_0(\mathbb{R})$, $(h * \mu)(0) = 0$, where $(h * \mu)(x) = \displaystyle\int_{-\infty}^{\infty} h(x-t)d\mu(t)$. Observe also that $h(x) \in C_0(\mathbb{R})$ if and only if $h(-x) \in C_0(\mathbb{R})$.

Assume that $\mu\hat{} = 0$. Then for all $f \in L^1(\mathbb{R})$, $f\hat{} * \mu(x) = (f * \mu\hat{})(x) = 0$. Hence, if we prove that $\{f\hat{} : f \in L^1(\mathbb{R})\}$ is dense in $C_0(\mathbb{R})$, then for each $h \in C_0(\mathbb{R})$ there is $f_n \in L^1(\mathbb{R})$ such that $f_n\hat{} \to h$ in $C_0(\mathbb{R})$. Since $f_n\hat{} * \mu(x) \to h * \mu(x)$ at each x, $h * \mu(x) = 0$.

To show that $\{f\hat{} : f \in L^1(\mathbb{R})\}$ is dense in $C_0(\mathbb{R})$, we let

$$F(x) = \frac{1}{\sqrt{2\pi}}(\frac{\sin(x/2)}{x/2})^2$$

and let $F_\rho(x) = \rho F(\rho x)$. Consider the integral

$$(h * F_\rho)(x) = \frac{2}{\pi\rho}\int_{-\infty}^{\infty} h(x-u)\frac{\sin^2(\rho u/2)}{u^2}du.$$

Define

$$\mathcal{F} = \{(h * F_\rho)(x) : h \in C_0(\mathbb{R})\bigcap L^1(\mathbb{R});\ \rho > 0\}.$$

Clearly, $\mathcal{F}$ is a subset of $C_0(\mathbb{R})\bigcap L^1(\mathbb{R})$ and is dense in $C_0(\mathbb{R})$.

Let $h \in C_0(\mathbb{R})\bigcap L^1(\mathbb{R})$. Then $(h * F_\rho)\hat{}(y) = h\hat{}(y)(F_\rho)\hat{}(y)$. Since $h \in L^1$, $h\hat{} \in C_0(\mathbb{R})$. Moreover,

$$F_\rho\hat{}(y) = \begin{cases} 1 - \frac{|y|}{\rho} & \text{if } |y| \le \rho \\ 0 & \text{if } |y| > \rho \end{cases}$$

belongs to $L^1(\mathbb{R})$. Therefore, $(h * F_\rho)\hat{} \in L^1(\mathbb{R})$. It follows from the inversion theorem that $h * F_\rho$ is the Fourier transform of a function in $L^1(\mathbb{R})$. Hence, $\mathcal{F}$ is a subset of $\{f\hat{} : f \in L^1(\mathbb{R})\}$. Since $\mathcal{F}$ is dense in $C_0(\mathbb{R})$, $\{f\hat{} : f \in L^1(\mathbb{R})\}$ is dense in $C_0(\mathbb{R})$. □

2. Kernels on $\mathbb{R}$

We define the Dirichlet, Fejér, and Poisson kernels on $\mathbb{R}$ by defining their Fourier transforms, see H. Helson [2].

$$\hat{D}_t(y) = \begin{cases} 1 & \text{if } |y| \le t \\ 0 & \text{if } |y| > t \end{cases}$$

$$\hat{K}_t(y) = \begin{cases} 1 - \frac{|y|}{t} & \text{if } |y| \le t \\ 0 & \text{if } |y| > t \end{cases}$$

and

$$\hat{P}_u(y) = e^{-u|y|}.$$

The parameters t and u are positive, having limits ∞ and 0^+, respectively.

Taking the inverse Fourier transform of $\hat{D}_t(y)$ we get the Dirichlet kernel

$$D_t(x) = \frac{\sin tx}{\pi x}.$$

Since $\hat{D}_t(y) \in L^1(\mathbb{R})$ and every point $y \neq t$ is a Lipschitz point of $\hat{D}_t(y)$, it follows from the inversion theorem that

$$\hat{D}_t(y) = \lim_{A,B\to\infty}\int_{-B}^{A} D_t(x)e^{-ixy}dx,\ y \neq \pm t.$$

That is, although $D_t(x)$ is not integrable, its Fourier transform in the generalized sense is $\hat{D}_t(y)$. Since $\hat{D}_t(y)$ is discontinuous, $D_t(x)$ cannot be integrable. Clearly, the Dirichlet kernel does not belong to the family of approximate identities.

To calculate the Fejér kernel, it follows from definitions that

$$\begin{aligned}(\hat{D}_t * \hat{D}_t)(y) &= \int_{-\infty}^{\infty} \hat{D}_t(y-\tau)\hat{D}_t(\tau)d\tau \\ &= \int_{-t}^{t} \hat{D}_t(y-\tau)d\tau \\ &= \int_{y-t}^{y+t} \hat{D}_t(u)du.\end{aligned}$$

To calculate the last integral, we consider two cases. If $|y| \geq 2t$, then the intervals $[y-t, y+t]$ and $[-t, t]$ are disjoint so that the integral equals zero; if $|y| < 2t$, then either $y+t$ or $y-t$ is in $(-t, t)$, but not both, so that the integral equals $2t - |y|$. Combining both results we get,

$(\hat{D}_t * \hat{D}_t)(y) =$

$$\int_{y-t}^{y+t} \hat{D}_t(u)du = \begin{cases} 2t - |y| & \text{if } |y| \leq 2t \\ 0 & \text{if } |y| > 2t \end{cases} = 2t\hat{K}_{2t}(y).$$

Also, by Theorem 1.14 we have that,

$$\begin{aligned}(D_t \cdot D_t)\hat{}(x) &= \frac{1}{2\pi}(\hat{D}_t * \hat{D}_t)(x) \\ &= \frac{1}{2\pi}(2t\hat{K}_{2t}(x)).\end{aligned}$$

Therefore, it follows from the inversion theorem that $(D_t \cdot D_t)(x) = \frac{1}{2\pi}(2tK_{2t}(x))$, or

$$2tK_{2t}(x) = \frac{1}{2\pi}(2\pi D_t(x))^2.$$

Hence, we obtain the Fejér kernel

$$K_t(x) = \frac{1}{2\pi t}(\frac{\sin(\frac{tx}{2})}{\frac{x}{2}})^2.$$

$K_t(x)$ is positive and integrable. Its Fourier transform is the function $\hat{K}_t(y)$ by the inversion theorem. Moreover, $\displaystyle\int K_t(x)dx = 1$ because $\hat{K}_t(y) = 1$ at $y = 0$. For any $\epsilon > 0$,

$$\int_{|x|>\epsilon} K_t(x)dx \leq \frac{1}{2\pi t}\int_{|x|>\epsilon} \frac{4}{x^2}dx \to 0$$

as $t \to \infty$. Hence (K_t) is an approximate identity on $\mathbb{R}$.

A direct computation of the inverse Fourier transform of $\hat{P}_u(y)$ gives

$$\begin{aligned}
P_u(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{P}_u(y)e^{ixy}dy \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-u|y|}e^{ixy}dy \\
&= \frac{1}{2\pi}\int_{-\infty}^{0}e^{uy}e^{ixy}dy+\frac{1}{2\pi}\int_{0}^{\infty}e^{-uy}e^{ixy}dy \\
&= \frac{1}{2\pi}\int_{-\infty}^{0}e^{y(u+ix)}dy+\frac{1}{2\pi}\int_{0}^{\infty}e^{-y(u-ix)}dy \\
&= \frac{1}{2\pi}(\frac{1}{u+ix}+\frac{1}{u-ix}) \\
&= \frac{u}{\pi(u^2+x^2)}.
\end{aligned}$$

This gives the formula for Poisson kernel

$$P_u(x)=\frac{u}{\pi(u^2+x^2)}.$$

Clearly, P_u is positive, and we check that

$$\lim_{u\to 0^+}\int_{-\epsilon}^{\epsilon}P_u(x)dx=1$$

for each $\epsilon>0$. Thus (P_u) is an approximate identity with $u\downarrow 0$.

Theorem 2.1 (Inversion Theorem)**.** *If f and $\hat{f}$ are both integrable, then $f(x)$ a.e. equals to a continuous function which is the inverse Fourier transform of $\hat{f}$, that is,*

$$f(x)=\frac{1}{2\pi}\int\hat{f}(y)e^{ixy}dy,\ a.e. \tag{2.1}$$

Proof: $P_u * f$ is continuous. We have,

$$\begin{aligned}
(P_u*f)(x) &= \int_{-\infty}^{\infty}P_u(x-t)f(t)dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}e^{-u|y|}e^{iy(x-t)}dy)f(t)dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-u|y|}e^{ixy}(\int_{-\infty}^{\infty}f(t)e^{-iyt}dt)dy \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-u|y|}e^{ixy}\hat{f}(y)dy.
\end{aligned}$$

Note that the P_u*f converges to $f(x)$ in $L^1(\mathbb{R})$ so that P_u*f converges to f almost everywhere at least on a subsequence of $u\downarrow 0$. We then obtain

$$f(x)=\lim_{u\downarrow 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-u|y|}e^{ixy}\hat{f}(y)dy=\frac{1}{2\pi}\int\hat{f}(y)e^{ixy}dy.$$

The last limit holds because of Lebesgue's dominated convergence theorem. □

Definition 2.1. *For any $f \in L^p(\mathbb{R}), 1 \le p < \infty$, we define the Poisson integral of f as*

$$F(x+iu) = P_u * f(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{uf(s)}{u^2+(x-s)^2}ds. \tag{2.2}$$

Since $P_u \in L^q(\mathbb{R})$, q conjugate exponent of p, $F(x+iu)$ is defined as a continuous function of x. [6] Moreover, $F(x+iu)$ provides a harmonic extension of f to the upper half plane. This can be verified directly.

Theorem 2.2. *The Poisson integral has a semigroup property: $P_u * P_v = P_{u+v}$ for all positive u and v.*

Proof: We have that

$$\begin{aligned}(P_u \hat{*} P_v)(y) &= \hat{P}_u(y)\cdot\hat{P}_v(y) = e^{-u|y|}\cdot e^{-v|y|}\\ &= e^{-(u+v)|y|} = \hat{P_{u+v}}(y).\end{aligned}$$

It follows from the inversion theorem that $P_u * P_v = P_{u+v}$. □

Theorem 2.3. *$||F(\cdot+iu)||_p$ increases as $u \downarrow 0$, for any $p, 1 \le p < \infty$. (if $p=1$, consider $P_u * \mu$). Similarly, if f is bounded, $\sup\limits_{x\in\mathbb{R}}|F(x+iu)|$ increases as $u \downarrow 0$.*

Proof: Let $v < u$ be given. Let $r = u - v \ge 0$. Then

$$||P_u * f||_p = ||P_{v+r} * f||_p = ||(P_r * P_v) * f||_p \le ||P_r||_1||P_v * f||_p = ||P_v * f||_p. \quad \square$$

Lemma 2.1. *Let $f_u(x) = F(x+iu)$ be a harmonic function in the upper half plane such that*

$$\sup_{u>0}||f_u(\cdot)||_p = A < \infty.$$

Then

$$f_{u+v}(x) = (P_u * f_v)(x).$$

Proof: $f_{u+v}(x) = (P_u * f_v)(x)$ says that the values of $F(u+ix)$ at the level $u+v$ are the values of $F(u+ix)$ at the level v convolved with the Poisson kernel with parameter u. [7]

We may assume that F is real. Fix $v > 0$. Define $G(x+iu) = P_u * f_v(x)$ (G is the Poisson integral of the values of F at level v). $G(x+iu)$ is harmonic in $u > 0$ and $\sup_{u>0}||G(\cdot,u)||_p \le ||f_v(\cdot)||_p < \infty$. Note that $G(x+iu)$ has boundary value (pointwise limit) $f_v(x)$ as $u \to 0$, which can be simply viewed as the value of $G(x+iu)$ when $u=0$. Therefore, $G(x+iu) - F(x+iu+iv)$ is a harmonic function in $u>0$, satisfying $\sup_{u>0}||G(\cdot+iu) - F(\cdot+iu+iv)||_p < \infty$, continuous on the closed upper half plane and null on the real axis $u=0$. Now, let

$$H(x+iu) = G(x+iu) - F(x+iu+iv).$$

We must show that $H(x+iu)$ vanishes for $u > 0$.

Let $h \in L^1(\mathbb{R})\bigcap L^q(\mathbb{R})$, where q is the conjugate exponent of p. Define

$$L(x+iu) = \int_{-\infty}^{\infty} h(x-y)H(y+iu)dy.$$

[6] If $f \in L^p(\mathbb{R})$, $1 \le p \le \infty$, and $g \in L^q(\mathbb{R})$, $\frac{1}{p}+\frac{1}{q}=1$, then $(f*g)(x)$ exists everywhere, belongs to $C(\mathbb{R})$, and $||f*g||_c \le ||f||_p||g||_q$.

[7] In periodic case, $P_r * f_s = f_{rs}$ is proved by using the fact that a harmonic function is the real part of an analytic function.

Then $L(x+iu)$ is continuous on the closed upper half plane, harmonic in the upper half plane, and is null on the real axis. Moreover, $\sup_{u>0} |L(x+iu)| \leq ||h||_q \sup_{u>0} ||H(\cdot+iu)||_p < \infty$, that is, $L(x+iu)$ is bounded on the upper half plane. Extend this function to a bounded harmonic function on the whole plane by setting $L(x-iu) = -L(x+iu)$ (Reflection Principle for Harmonic Functions). By Liouville's theorem, L is constant. Letting h range over an approximate identity shows that H is a constant, and since it vanishes on the real axis, is null. □

Theorem 2.4. *Let $f_u(x) = F(x+iu)$ be a harmonic function in the upper half plane. Then there exists a $f \in L^p(\mathbb{R})$, $1 < p \leq \infty$, so that $f_u(x) = P_u * f(x)$ if and only if $f_u(x) \in L^p(\mathbb{R})$ with the norm bounded by a constant independent of $u > 0$, that is,*

$$\sup_{u>0} A_u = \sup_{u>0} ||f_u(\cdot)||_p = A < \infty.$$

Proof: *Necessity:* If we think of $P_u * f(x)$ as a family (with continuous parameter $u > 0$) of functions $f_u(x)$ defined on $\mathbb{R}$, then as $p > 1$,

$$||f_u||_p = ||P_u * f||_p \leq ||P_u||_1 ||f||_p.$$

Hence, $\{f_u\},\ u > 0$, is bounded in $L^p(\mathbb{R})$.

Sufficiency: Assume that $f_u(x) = F(x+iu)$ is bounded in $L^p(\mathbb{R})$. If $1 < p \leq \infty$, by Banach-Alaoglu's theorem (view $L^p(\mathbb{R})$ as the dual of separated normed space $L^q(\mathbb{R})$, $1 \leq q < \infty$), $\{f_u\}$ is weakly* sequentially compact in $L^p(\mathbb{R})$, that is, there is an element f of $L^p(\mathbb{R})$ such that every *-neighborhood of f contains f_u for arbitrary small positive u. In other words, there is a subsequence f_{v_j} of f_u that is weakly* convergent to *some* $f \in L^p(\mathbb{R})$ as $v_j \downarrow 0^+$, i.e., for all $g \in L^q(\mathbb{R})$, $\displaystyle\int f_{v_j} g \to \int fg$ as $v_j \downarrow 0^+$. In particular, since for each x, $P_u(x-t) \in L^q(\mathbb{R})$, $1 \leq q < \infty$, we have $P_u * f_{v_j}(x) = \displaystyle\int P_u(x-t) f_{v_j}(t)dt$ tends to $\displaystyle\int P_u(x-t)f(t)dt = P_u * f(x)$ as $v_j \downarrow 0^+$. On the other hand, $P_u * f_{v_j}(x) = f_{u+v_j}(x)$ (see Lemma 2.1), which converges to $f_u(x)$ by the continuity of $F(x+iu)$. Therefore, $P_u * f(x) = f_u(x)$ for all x.

If $1 < p < \infty$, then $P_u * f \to f$ in the norm of $L^p(\mathbb{R})$ (Fejer's theorem). If $p = \infty$ then $P_u * f \to f$ in weak* topology in $L^\infty(\mathbb{R})$, i.e., for every $s(x) \in L^1(\mathbb{R})$, $\lim_{u\downarrow 0^+} \int [P_u * f(x) - f(x)]s(x)dx = 0$. (For a proof, see Butzer [1].) □

Theorem 2.5. *Let $f_u(x) = F(x+iu)$ be a function harmonic in the upper plane $u > 0$. Then there is a unique measure $\mu \in M(\mathbb{R})$ such that*

$$f_u(x) = F(x+iu) = P_r * \mu(x) = \int_{-\infty}^{\infty} P_u(x-t)d\mu(t)$$

if and only if

$$A_u = \int |F(x+iu)|dx \leq K, \quad \forall \quad u > 0.$$

Moreover, $||\mu|| = \lim_{u\downarrow 0} A_u$.

Proof: *Necessity:* If we think of $P_u * \mu(x)$ as a family (with continuous parameter $u > 0$) of functions defined on $\mathbb{R}$, then

$$||f_u||_1 = ||P_u * \mu|| \leq ||P_u||_1 ||\mu||.$$

Therefore, $\{f_u\}, u > 0$, is bounded in $L^1(\mathbb{R})$.

Sufficiency: By assumption, $||f_u||_1 \leq K$, i.e., $||f_u(x)dx||_{M(\mathbb{R})} = ||f_u||_1 \leq K, \quad \forall u > 0$. Since $C_0(\mathbb{R})$, as the pre-dual of $M(\mathbb{R})$, is separable normed space, by Banach-Alaoglu theorem the closure of $\{f_u(x)dx\}$ in $M(\mathbb{R})$ is weak* sequentially compact. Therefore, there is a subsequence $\{f_{v_j}\}(x)dx$ of $f_u(x)dx$ that converges to some $\mu \in M(\mathbb{R})$ in weak* topology. That is,

$$\int h(e^{-it})f_{v_j}(t)dt \to \int h(e^{-it})d\mu(t), \qquad v_j \to 0$$

for each $h \in C_0(\mathbb{R})$. In particular, since for each x, $P_u(x-t) \in C_0(\mathbb{R})$,

$$\int P_u(x-t)f_{v_j}(t)dt \to \int P_u(x-t)d\mu(t),\ v_j \to 0.$$

On the other hand,

$$P_u * f_{v_j}(x) = f_{u+v_j}(x) \to f_u(x),\ v_j \to 0.$$

Hence, $f_u(x) = \displaystyle\int P_u(x-t)d\mu(t)$ for all x.

We show that $||\mu|| = \lim_{u\downarrow 0} A_u$. Note that $\mu = \lim_{j\to\infty} f_{v_j}(x)dx$ in the weak* topology of $M(\mathbb{R})$ as the dual of $C_0(\mathbb{R})$. It follows that $||\mu|| \leq \liminf_{j\to\infty} A_{v_j}$ where $A_{v_j} = ||f_{v_j}||_1$ (For a proof, see the Appendix). Since A_u increases with $u \downarrow$ and $A_u \leq K$, $||\mu|| \leq \lim_{u\to 0} A_u$. Furthermore, the inequality cannot be strict. Note that $f_u = P_u * \mu$ and $||f_u||_1 \leq ||P_u||_1||\mu||$. Therefore, $A_u = ||f_u||_1 \leq ||\mu||$ for every $u > 0$. If the inequality were strict, we would have $A_u \leq ||\mu|| < \lim_{u\to 0} A_u$ for $u > 0$, which is impossible.

As to the norm convergence of $||f_u - \mu||_{M(\mathbb{R})} \to 0$ as $u \to 0$, if μ is absolutely continuous then $\mu = f(x)dx$ for some $f \in L^1(\mathbb{R})$. Hence $f_u = P_u * \mu$ is indeed $f_u = P_u * f$. Thus, by Fejer's theorem, $||f_u - f||_1 \to 0$. That is, $||f_u - \mu||_{M(\mathbb{R})} \to 0$ as $u \to 0$. □

3. The Plancherel Theorem

In this section we define

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ixy}dx.$$

Lemma 3.1. *Let $\mathcal{C}$ be the collection of continuously differentiable functions with compact support. Then $\mathcal{C} \subset L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$ and $\mathcal{C}$ is a dense subspace of $L^2(\mathbb{R})$.*

Proof: Let $f \in L^2(\mathbb{R})$. Define $f_k(x) = f(x)$ if $|x| \leq k$; and $f_k(x) = 0$ if $|x| > 0$. Then $f_k \to f$ in $L^2(\mathbb{R})$. Furthermore, we may choose an approximate identity with compact support and continuous derivative, for instance, let $h(x) = e^{-\frac{1}{x^2}}$ for $x \geq 0$ and $h(x) = 0$ for $x < 0$. Then $h \in C^\infty(\mathbb{R})$ and $\phi(x) = h(x+1)h(1-x) \in C^\infty(\mathbb{R})$ and has compact support $[-1, 1]$, and $\int \phi(x)dx = 1$, when properly normalized. Let $e_n(x) = n\phi(nx)$. Then $e_n(x)$ is an approximate identity with compact support and continuous derivative (in fact, $C^\infty(\mathbb{R})$). Since for each k, f_k has compact support, $e_n * f_k$ provides a continuously differentiable approximation with compact support to f_k in $L^2(\mathbb{R})$. Hence $\mathcal{C}$ is dense in $L^2(\mathbb{R})$.

Lemma 3.2. *If $f \in \mathcal{C}$, then $\hat{f} \in L^2(\mathbb{R})$. Moreover, $||\hat{f}||_2 = ||f||_2$. Hence, the Fourier transform $\hat{f}$ (as defined in this section) is isometric from $\mathcal{C}$ to $\mathcal{F}(\mathcal{C})$ as subspaces of $L^2(\mathbb{R})$.*

Proof: Let $f \in \mathcal{C}$. Define $\tilde{f}(x) = \overline{f(-x)}$. Then $f * \tilde{f}(x) \in \mathcal{C}$. By the inversion theorem, at every point x where $(f * \tilde{f})(x)$ satisfies the Lipschitz condition, we have

$$(f * \tilde{f})(x) = \lim_{A,B\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-B}^{A} \widehat{f * \tilde{f}}(y) e^{ixy} dy.$$

Since $f * \tilde{f}(x) \in \mathcal{C}$, it satisfies the Lipschitz condition at *every* point, in particular, at $x = 0$, we have

$$(f * \tilde{f})(0) = \lim_{A,B\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-B}^{A} \widehat{f * \tilde{f}}(y) dy.$$

Note that $||f||_2 = (f * \tilde{f})(0)$, $\widehat{\tilde{f}} = \overline{\hat{f}}$, and $\dfrac{1}{\sqrt{2\pi}} \widehat{f * \tilde{f}}(y) = |\hat{f}(y)|^2$. We have $||f||_2 = ||\hat{f}||_2$.

The Fourier transform is an isometry defined on $\mathcal{C}$. Since it is defined on a dense subspace of $L^2(\mathbb{R})$, it has a unique *continuous* extension to an isometry $\mathcal{F}$ of all of $L^2(\mathbb{R})$ *into* itself, which is defined as follows: for $f \in L^2(\mathbb{R})$, let $f_n \in \mathcal{C}$ such that $f_n \to f$. Since $||\mathcal{F}f||_2 = ||f||_2$ for all $f \in \mathcal{C}$, $\hat{f}_n$ is a Cauchy sequence in $L^2(\mathbb{R})$ and so converges to *some* $g \in L^2(\mathbb{R})$. We define $\mathcal{F}(f) = g$. Let us show $||\mathcal{F}(f)||_2 = ||f||_2$ for all $f \in L^2(\mathbb{R})$. Let $f \in L^2(\mathbb{R})$ and $f_k \in \mathcal{C} \to f$. Then by definition of $\mathcal{F}$, $||\mathcal{F}f_k||_2 \to ||\mathcal{F}f||$. On the other hand, $||\mathcal{F}f_k|| = ||f_k||_2 \to ||f||_2$. Therefore, $\mathcal{F}$ is an isometry of $L^2(\mathbb{R})$ *into* $L^2(\mathbb{R})$. We will prove that $\mathcal{F}$ is indeed 'onto'. □

Lemma 3.3. *The Fourier transform of* $L^2(\mathbb{R})$ *is onto, i.e.,* $E = \{\mathcal{F}(f) : f \in L^2(\mathbb{R})\} = L^2(\mathbb{R})$.

Proof: First we prove that E is dense in $L^2(\mathbb{R})$.

We prove that for each $h \in \mathcal{C}$, $< \mathcal{F}f, h > = < f, h^* >$ for all $f \in L^2(\mathbb{R})$, where h^* is defined by the formula:

$$h^*(x) = \frac{1}{\sqrt{2\pi}} \int h(y) e^{ixy} dy.$$

In fact, for $f, h \in \mathcal{C}$ we have

$$\int \hat{f}\overline{h} = \int f\overline{h^*},$$

that is, $< \mathcal{F}f, h > = < f, h^* >$ for all $f \in \mathcal{C}$. It follows that $< \mathcal{F}f, h > = < f, h^* >$ for all $f \in L^2(\mathbb{R})$.

The operator $\mathcal{F}^*$ defined on $L^2(\mathbb{R})$ by $\mathcal{F}^*h = h^*$ is called the *adjoint operator* of $\mathcal{F}$. (see the Appendix). Note that $\mathcal{F}$ is essentially the Fourier transform, and therefore, is an isometry. Thus its null space $N(\mathcal{F}^*)$ contains 0 only (uniqueness theorem for F.T.). Since $N(T^*)^\perp = R(T)$ (see the Appendix), it follows that E, the range of $\mathcal{F}$, is dense in $L^2(\mathbb{R})$.

To prove $E = L^2(\mathbb{R})$, we show that E is closed. Take $g \in \overline{E}$. Then there exists $g_k \in E$ with $g_k \to g$. Let f_k be such that $\mathcal{F}f_k = g_k$. Since $\mathcal{F}$ is an isometry, f_k is a Cauchy sequence converging to some $f \in L^2(\mathbb{R})$ and we must have $\mathcal{F}f = g$. Since $\overline{E} = L^2(\mathbb{R})$ and E is closed, $E = L^2(\mathbb{R})$. □

Theorem 3.1 (Plancherel)**.** *The Fourier transform* $\mathcal{F}$ *is a unitary operator on* $L^2(\mathbb{R})$ *and the inverse Fourier transform,* $\mathcal{F}^{-1}$*, can be obtained by* $(\mathcal{F}^{-1}f)(x) = (\mathcal{F}f)(-x)$ *for all* $f \in L^2(\mathbb{R})$.

Proof: Since $\mathcal{F}$ is an isometry of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$, $\mathcal{F}$ is a unitary operator on $L^2(\mathbb{R})$. It follows from the properties of a unitary operator that $\mathcal{F}^{-1} = \mathcal{F}^*$ (see the Appendix). The form of $\mathcal{F}^*$ can be easily found when acting on $f \in \mathcal{C}$: Let $g \in \mathcal{C}$. A change of order of integration gives

$$\int (\mathcal{F}g)(x)\overline{f(x)}dx = \int g(x)\overline{f^*(x)}dx,$$

i.e., $< \mathcal{F}(g), f >=< g, f^* >$ for all $g \in \mathcal{C}$, where $f^* = \mathcal{F}^*(f)$ is defined by the formula:

$$f^*(x) = \frac{1}{\sqrt{2\pi}} \int_R f(y)e^{ixy}dy.$$

It follows that $(\mathcal{F}^{-1}f)(x) = (\mathcal{F}f)(-x)$ for all $f \in \mathcal{C}$. For $f \in L^2(\mathbb{R})$, we take $f_k \in \mathcal{C}$ with $f_k \to f$ in $L^2(\mathbb{R})$. Then

$$(\mathcal{F}^{-1}f)(x) = l.i.m.(\mathcal{F}^{-1}f_k)(x) = l.i.m.(\mathcal{F}f_k)(-x) = (\mathcal{F}f)(-x).$$

This shows that $(\mathcal{F}^{-1}f)(x) = (\mathcal{F}f)(-x)$ for all $f \in L^2(\mathbb{R})$. □

Lemma 3.4 (Multiplication Formula). *If $f, g \in L^1(\mathbb{R})$, then*

$$\int \hat{f}g = \int f\hat{g}. \tag{3.1}$$

Proof: Since $F(x,t) = f(t)g(x)e^{-itx}$ is a measurable function on $\mathbb{R}\times\mathbb{R}$ and $||F(x,t)||_{L(\mathbb{R}^2)} = ||f||_1||g||_1$, we can apply Fubini's theorem to obtain

$$\int \hat{f}(t)g(t)dt = \int f(t)(\int g(x)e^{-ixt}dx)dt = \int f(t)\hat{g}(t)dt. \quad \square$$

As an application of the multiplication formula, we prove the following Fourier inverse theorem.

Lemma 3.5. *If $f \in L^1(\mathbb{R})$, then the Abel mean of the Fourier integral converges to $f(x)$ a.e., i.e., for almost every x,*

$$\lim_{\epsilon\to 0} \frac{1}{2\pi} \int \hat{f}(t)e^{-\epsilon|t|}e^{ixt}dt = f(x). \tag{3.2}$$

Proof: Let

$$g(t) = e^{-\epsilon|t|}e^{ixt}, \ \epsilon > 0.$$

Then

$$\hat{g}(t) = 2\pi P_\epsilon(x-t) = \frac{2\epsilon}{\epsilon^2 + (x-t)^2}.$$

Using the multiplication formula, we get, if $f \in L^1(\mathbb{R})$,

$$\frac{1}{2\pi}\int \hat{f}(t)e^{-\epsilon|t|}e^{ixt}dt = \int f(t)P_\epsilon(x-t)dt.$$

Since the latter convolution converges to $f(x)$ a.e., [8]

$$\lim_{\epsilon\to 0} \frac{1}{2\pi} \int \hat{f}(t)e^{-\epsilon|t|}e^{ixt}dt = f(x). \quad \square$$

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[8] At every point of x for which

$$\int_0^h [f(x+u) + f(x-u) - 2f(u)]du = o(h),$$

$\lim f(t)P_\epsilon(x-t)dt \to f(x)$.

Corollary 3.1 (Inversion Theorem)**.** *If $f \in L^1(\mathbb{R})$ so that $\hat{f} \in L^1(\mathbb{R})$, then for a.e. x,*

$$\frac{1}{2\pi}\int \hat{f}(t)e^{ixt}dt = f(x). \tag{3.3}$$

In particular, the inversion formula holds at every x for which

$$\int_0^h [f(x+u)+f(x-u)-2f(u)]du = o(h)$$

holds.

Proof: If $\hat{f} \in L^1(\mathbb{R})$, the corollary follows from Lemma 3.5 by applying the Lebesgue Dominated Convergence Theorem. □

Lemma 3.6. *If $f(x) \in L^1(\mathbb{R})$ is continuous at $x = 0$ such that $\hat{f} \geq 0$, then $\hat{f} \in L^1(\mathbb{R})$ and*

$$f(x) = \frac{1}{2\pi}\int \hat{f}(y)e^{ixy}dy, a.e.$$

In particular,

$$f(0) = \frac{1}{2\pi}\int \hat{f}(y)dy.$$

Proof: We need only to show that $\hat{f} \in L^1(\mathbb{R})$. Then the rest of the statements follows from the inversion theorem.

Note that Corollary 3.1 holds at every point x for which

$$\int_0^h [f(x+u)+f(x-u)-2f(u)]du = o(h).$$

In particular, it holds at the point $x = 0$ of continuity of f, i.e.,

$$\frac{1}{2\pi}\int \hat{f}(t)e^{-\epsilon|t|}dt = f(0).$$

By Fatou's lemma,

$$\int \hat{f}(y)dy \leq \lim_{\epsilon\to 0}\int \hat{f}(t)e^{-\epsilon|t|}dt = f(0).$$

Since $0 \leq \hat{f}$, $\hat{f} \in L^2(\mathbb{R})$. □

Lemma 3.7. *If $f \in L^1(\mathbb{R})\bigcap L^2(\mathbb{R})$, then $\hat{f} \in L^2(\mathbb{R})$ and $||\hat{f}||_2 = ||f||_2$.*

Proof: Define $\tilde{f}(x) = \overline{f(-x)}$. Since $f, \tilde{f} \in L^2(\mathbb{R})$, $h = f * \tilde{f} \in L^1(\mathbb{R})$ and is continuous. Further, $\hat{h}(y) = |\hat{f}(y)|^2 \geq 0$. Hence, $\hat{h} \in L^1(\mathbb{R})$ and $h(0) = \int \hat{h}(y)dy$. It follows that

$$\int |\hat{f}(y)|^2 dy = \int \hat{h}(y)dy = h(0) = f * \tilde{f}(0) = \int |f(x)|^2 dx. \quad \square$$

Since the Fourier transform is an isometry of $L^1(\mathbb{R})\bigcap L^2(\mathbb{R})$, it has a unique *continuous* extension to an isometry $\mathcal{F}$ of all of $L^2(\mathbb{R})$ *into* itself with $||\mathcal{F}f||_2 = ||f||_2$ for all $f \in L^2(\mathbb{R})$. For $f \in L^2(\mathbb{R})$, define $f_n(x) = f(x)$ if $|x| \leq n$ and $f_n(x) = 0$ if $|x| > n$. Then $f_n \in L^1(\mathbb{R})\bigcap L^2(\mathbb{R})$ and $f_n \to f$ in $L^2(\mathbb{R})$. Define $\mathcal{F}f = l.i.m.\hat{f}_n$. We'll prove that $\mathcal{F}$ is indeed 'onto'.

Lemma 3.8 (Multiplication Formula for $L^2(\mathbb{R})$)**.** *If $f, g \in L^2((\mathbb{R})$ then*

$$\int \hat{f} g = \int f \hat{g}. \tag{3.4}$$

Proof: Fix $g \in L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$ first. Let $f \in L^2(\mathbb{R})$ and $f_k \in L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$ with $l.i.m.f_k = f$. Since $\hat{g} \in L^2(\mathbb{R})$, $\displaystyle\int f_k \hat{g} \to \int f \hat{g}$. It follows from the multiplication formula for $L^1(\mathbb{R})$ that $\displaystyle\int f_k \hat{g} = \int \hat{f}_k g \to \int \hat{f} g$. Hence for $f \in L^2(\mathbb{R})$ and $g \in L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$, $\displaystyle\int \hat{f} g = \int f \hat{g}$. Starting with this formula, for $f, g \in L^2(\mathbb{R})$, we approximate g by $g_k \in L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$. □

Theorem 3.2 (Plancherel)**.** *The Fourier transform $\mathcal{F}$ is a unitary operator of $L^2(\mathbb{R})$ and the inverse Fourier transform, $\mathcal{F}^{-1}$, can be obtained by $(\mathcal{F}^{-1} f)(x) = (\mathcal{F} f)(-x)$ for all $f \in L^2(\mathbb{R})$.*

Proof: We have already proved that $\mathcal{F}$ is an isometry, we only need to show $\mathcal{F}$ maps $L^2(\mathbb{R})$ *onto* $L^2(\mathbb{R})$, i.e., $E = \{\mathcal{F}(f) : f \in L^2(\mathbb{R})\} = L^2(\mathbb{R})$. As proven before, E is closed. Assume that $E \neq L^2(\mathbb{R})$. Then there exists $g \neq 0, g \in L^2(\mathbb{R}) \setminus E$, such that $< g, f >= 0$ for all $f \in E$, or $< g, \hat{h} >= 0$ for all $h \in L^2(\mathbb{R})$. It follows from the multiplication formula that $\displaystyle\int h\overline{\hat{g}} = 0$ for all $h \in L^2(\mathbb{R})$. In particular, taking $h = \hat{g} \in L^2(\mathbb{R})$, $||\hat{g}||_2 = 0 = ||g||_2$ and $g = 0$ a.e., contrary to the assumption $g \neq 0$. Therefore, $\mathcal{F}$ is onto and so is a unitary operator of $L^2(\mathbb{R})$. □

4. Appendix

4.1. **Weak/Weak * Topologies in Linear Spaces.** Let X be a topological linear space and X' be its conjugate space of all continuous linear functionals on X. [9]

The *weak topology* $\sigma(X, X')$ on X is defined as follows:
Let F be a nonempty finite subset of X'. Define

$$p_F(x) = max_{x' \in F} |x'(x)|, \quad x \in X.$$

$p_F(x)$ is a seminorm on X. $\sigma(X, X')$ is the locally convex topology on X defined by the family of all seminorms $p_F(x)$, where F ranges over all finite subsets of X'. A base at $x_0 \in X$ for this topology is given by sets of the form

$$\begin{aligned} U_{F,r} &= \{x : |x'(x) - x'(x_0)| < r \quad \text{for each } x' \in F\} \\ &= \bigcap_{x' \in F} \{x : |x'(x) - x'(x_0)| < r\}, \end{aligned}$$

where $r > 0$ and F is a nonempty finite subset of X'. $\sigma(X, X')$ is the weakest topology on X for which all the elements of X' are continuous.

[9] When X is a Hausdorff locally convex space, the Hahn-Banach theorem ensures the existence of enough elements in X' to make possible a rich theory of the duality between X and X'.

A sequence $\{x_n\}$ in a normed linear space X converges to an element $f \in X'$ in weak topology if and only if $\lim_{n\to\infty} f(x_n) = f(x)$ for all $f \in X'$.

The *weak* topology* $\sigma(X', X)$ on X' is defined as follows:
Let A be a nonempty finite subset of X. Define

$$p_A(x') = max_{x\in A}|x'(x)|, \quad x' \in X'.$$

$p_A(x')$ is a seminorm on X'. $\sigma(X', X)$ is the locally convex topology on X' defined by the family of all seminorms $p_A(x')$, where A ranges over all finite subsets of X. A base at $x'_0 \in X'$ for this topology is given by sets of the form

$$\begin{aligned} U_{A,r} &= \{x' : |x'(x) - x'_0(x)| < r \quad \text{for each } x \in A\} \\ &= \bigcap_{x\in A} \{x : |x'(x) - x'_0(x)| < r\}, \end{aligned}$$

where $r > 0$ and A is a nonempty finite subset of X. $\sigma(X', X)$ is the weakest topology on X' for which $x'(x)$, as a linear functional acting on X', is continuous.

A sequence $\{f_n\}$ in X' of a normed linear space X converges to an element $f \in X'$ in weak* topology if and only if at each x, $\lim_{n\to\infty} f_n(x) = f(x)$, see K. Yosida [4].

Theorem 4.1. *If X is a Banach space, then $\{f_n\} \subset X'$ converges weakly* to an element $f \in X'$ if and only if (1). $\{||f_n||\}$ is bounded; and (2). $\lim f_n(x) = f(x)$ for all x in a dense subset (with respect to norm topology) of X.*

Proposition 4.1. *In $X = l^2$, for $1 \le m < n < \infty$, let $x_{mn} \in l^2$ be defined as*

$$x^{(k)}_{mn} = \begin{cases} 1 & \text{if } k = m \\ m & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

and let $A = \{x_{mn} : 1 \le m < n < \infty\}$. Then no sequence of elements of A converges weakly to the origin, yet the origin is an accumulation point of A in the weak topology.

Proof: Note that l^2 is reflexible, so the weak topology and the weak* topology coincide on l^2. To prove that $0 \in l^2$ is an accumulation point of A in the weak topology, i.e., to prove that, since $0 \notin A$, for any weak neighborhood S of 0, $S \bigcap A$ is not empty, [10] we have to be able to write down a base at 0 :

$$U_F = \bigcap_{b\in F} \{a \in l^2 : | < a, b > | = | \sum a^{(k)}\overline{b^{(k)}}| < \epsilon\},$$

where F is a finite subset of l^2 and $\epsilon > 0$. In particular, for every fixed $b \in l^2$,

$$U_{b,\epsilon}(0) = \{a \in l^2 : | < a, b > | < \epsilon\}$$

is a weak neighborhood of 0.

[10]If $A \subset X$, then $x_0 \in X$ is called an **accumulation point** of A if every neighborhood of x_0 contains a point of $A \setminus \{x_0\}$. If A is a subset of a Hausdorff space X and x_0 is an accumulation point of A, then every neighborhood of x_0 contains **infinitely** many points of A. The **closure** of A consists of points x such that every neighborhood of x contains at least a point of A.

Given a weak neighborhood $U = U_{b,\epsilon}(0)$ of 0, can we always find $x_{mn} \in A$ so that $x_{mn} \in U$? Observe that $| < x_{mn}, b > | = |b^{(m)} + mb^{(n)}|$, which can be made as small as we wish. First we choose m large enough so that $|b^{(m)}|$ is very small, then for this fixed m, choose n large enough so that $|mb^{(n)}|$ is also very small.

Can we prove that there is no sequence of elements in A that converges weakly to 0? Given any sequence of elements in A, we show that there exist $\epsilon_0 > 0$ and $b \in l^2$ (i.e. there exists a weak neighborhood $U_{b,\epsilon_0}(0)$ of 0) such that for any l, we can always find an element a in this sequence with subscript $\geq l$ such that $a \notin U_{b,\epsilon_0}(0)$.

Consider a sequence, ξ, of elements of $x_{mn} \in A$. If *some integer,* say l, appears infinitely many times as the m-index of $x_{mn} \in \xi$, then we choose b so that $b^{(l)} = 1$, $b^{(k)} = 0$ $k \neq l$. Of course, $b \in l^2$ and there is a (of course, infinite) subsequence $\{x_{ln}\}$ of ξ with $| < b, x_{ln} > | = 1$. If none of the integers appears infinitely many times as m-index in ξ, then the range of m-index of elements $x_{mn} \in \xi$ is unbounded. We may extract a subsequence, call it η, of ξ so that their m-indices form a (strictly) increasing sequence. Note that the range of n-index of $x_{mn} \in \eta$ is unbounded. We may extract a further subsequence, call it ζ, of η so that their n-indices form a (strictly) increasing sequence of integers. Now we define b with $b^{(n)} = \frac{1}{m}$ if $x_{mn} \in \zeta$ (Note: for each n there is only one m such that $x_{mn} \in \zeta$) and $b^{(n)} = 0$, otherwise. Note that if $x_{mn} \in \zeta$, then $| < x_{mn}, b > | = 1$. All that remains is to notice that $b \in l^2$.

Definition 4.1. *Let X be a topological space. If $A \subset X$ is such that every sequence in A has a subsequence that converges to a point in A, then A is called* ***sequentially compact.***

Theorem 4.2. *Let X be a normed linear space. If $F \subset X'$ is weak* sequentially compact, then F is countably weak* compact.*

Proof: Suppose that there is an open cover (in weak* topology) $\{U_j\}$ of F for which there is no finite subcover. Then for any finite collection U_j, $1 \leq j \leq n$, $F \setminus \bigcup_{j=1}^{n} U_j \neq \phi$. Pick $x_1 \in F \setminus U_1$; suppose $x_1 \in U_{n_1}$. Then pick $x_2 \in F \setminus (U_1 \bigcup \cdots U_{n_1})$. Suppose x_k has been chosen and $x_k \in U_{n_k}$. Choose $x_{k+1} \in F \setminus (U_1 \bigcup \cdots U_{n_k})$. These points $\{x_k\}$ must all be distinct. The sequence $\{x_k\}$ has a subsequence that converges weak* sequentially to a point $y \in F$. We assume that $y \in$ (*some*) U_n.

Now let k' be such that $n_{k'} > n$. Then $x_k \notin U_n$ for all $k > k'$. Hence U_n contains only finitely many points of $\{x_k\}$ and so the subsequence we found above cannot converge to y in weak* topology. This is a contradiction. □

Theorem 4.3 (Banach-Alaoglu)**.** *If X is a normed space then $S* = \{x' \in X' : ||x'|| \leq 1\}$ is weak* compact.*

Theorem 4.4. *If (X, T) is compact and if there exist continuous functions $\{f_n : X \to \mathbb{R}\}$ that separate points in X (i.e. for any $x, y \in X$, $x \neq y$, there is n such that $f_n(x) \neq f(y)$), then (X, T) is metrizable.*

Theorem 4.5. *If X is a separable normed linear space and $K \subset X'$ is weak* compact, then $(K, W*)$ is metrizable.*

Proof: By the above theorem we need only to find a countable family of continuous functions from $(K, W*)$ to $\mathbb{R}$ which separates points in K. Let $x_n \in X$ and $\{x_n\}$ be dense in X. Let $\Lambda_n : X' \to C$ be defined as $\Lambda_n(x') = x'(x_n)$. Then each Λ_n is $W*$ continuous (by definition of weak* topology). Also $\{\Lambda_n\}$ separates points in K. In fact, if $x' \neq y'$ are two elements in X' and $\Lambda(x') = \Lambda_n(y')$ for all n, then x' and y' coincide on a dense subset of X and $x' = y'$. A contradiction.

Theorem 4.6 (Weak* Compactness Theorem)**.** *If X is a separable normed linear space then the bounded sets in X' are weak* conditionally sequentially compact. That is, if X is separable and $x'_n \in X'$ with $||x'_n|| \leq A$, then there is $x'_0 \in X'$ with $||x'_0|| \leq A$ and a subsequence x'_{n_k} such that $x'_{n_k} \to x'_0$ in weak* topology, i.e., at each $x \in X$, $x'_{n_k}(x) \to x'(x)$ as $k \to \infty$. (cf. page 22. Butzer)*

Proof: The proof is obtained by putting together the Banach-Alaoglu theorem and the above theorem. □

Corollary 4.1 (Weak* Compactness Theorem for $L^p(\mathbb{R})$, $1 < p \leq \infty$.)**.** *For $1 < p \leq \infty$, if $||f_n||_p \leq A$ then there is f_0, $||f_0||_p \leq A$ and $\{f_{n_k}\}$ so that for each $g \in L^{p'}$, $\displaystyle\int f_{n_k} g \to \int f_0 g$.*

Proof: $L^p(\mathbb{R})$, $1 < p \leq \infty$, are conjugate spaces of $L^{p'}(\mathbb{R})$, $1 \leq p < \infty$, which are separable. □

Corollary 4.2. *Let $\mu_n \in M$ (all finite Borel measures on $\mathbb{R}^n$) be such that $||\mu_n||_M \leq A$ for all n. Then there is $n_k \to \infty$ and $\mu \in M$ so that $\mu_{n_k} \to \mu$ in weak* topology on M, that is, for any $f \in C_0(\mathbb{R}^n)$, $\displaystyle\int f d\mu_{n_k} \to \int f d\mu$. Moreover,*

$$||\mu|| \leq \liminf_{k\to\infty} ||u_{n_k}||.$$

Proof: $M(\mathbb{R}^n)$ is the conjugate space of $C_0(\mathbb{R}^n)$ (by Riesz's theorem) which is separable. □

Corollary 4.3 (Weak* Compactness Theorem for $L^1(\mathbb{R})$)**.** *Let $f_n \in L^1(\mathbb{R})$ such that $||f_n||_1 \leq K$ for all n. Then there exist a subsequence f_{n_k} and $\mu \in M(\mathbb{R})$ such that*

$$\lim_{k\to\infty} \int_{\mathbb{R}} f_{n_k}(x) g(x) dx = \int_{\mathbb{R}} g(x) d\mu(x)$$

for each $g \in C_0(\mathbb{R})$.

Proof: We may view each f_n as an element of $M(\mathbb{R})$, if we identify $f_n(x)$ with $f_n(x)dx$. Moreover, $||f_n(x)dx||_{M(\mathbb{R})} = ||f_n||_1 \leq K$ for all n. The corollary follows. □

4.2. **Dual or Conjugate Operators and Adjoint Operators.** Let X, Y be Locally convex linear topological spaces. Let T be a linear operator on $D(T) \subset X$ into Y. Let $\{x', y'\}$ be a point in $X' \times Y'$ satisfying

$$< Tx, y' > = < x, x' > \ \forall x \in D(T).$$

Then x' is determined uniquely by y' iff $D(T)$ is dense in X.

In this case, a linear operator T' defined by $T'y' = x'$ is called the *dual or conjugate operator* of T. Its domain is the set of all $y' \in Y'$ such that there exists $x' \in X'$ satisfying $< Tx, y' >=< x, x' >$ for all $x \in D(T)$.

Let X and Y be complex Hilbert spaces. Let E_X be the operator that associates to each $y \in X$ the linear functional $x \to< x, y >$. Then E_X is a 'conjugate-linear' isometry of X onto X. Let T be a *densely defined linear operator* from X into Y. The *adjoint* operator of T is the operator T^* defined by

$$T^* = E_X^{-1} T' E_Y,$$

where the domain of T^* is the set of all y for which $(E_X^{-1} T' E_Y)(y)$ is defined.

The notion of transposed conjugate matrix may be extended to the notion of adjoint operator in Hilbert spaces. In contrast, the notion of transposed matrix may be extended to the notion of dual operator in locally convex linear topological spaces.

Clearly,

$$D(T^*) = \{y : E_Y(y) \in D(T')\} = \{y : x \to< Tx, y > \text{ is continuous on } D(T)\}.$$

One can show that $y \in D(T^*)$ if and only if there exists a $y^* \in X$ such that

$$< Tx, y >=< x, y^* >$$

holds for all $x \in D(T)$. In this case, $y^* = T^*y$. If $T \in L(X, Y)$, then $T^* \in L(Y, X)$ and $||T^*|| = ||T||$. In general, if $\overline{D(T)} = X$, then T^* is a closed linear operator.

It is known that $\overline{R(T)} = N(T^*)^\perp$. We include a proof. If $y \in \overline{R(T)}$, then there exist $x_n \in D(T)$ such that $Tx_n \to y$. Take $z \in N(T^*)$. Then $< Tx_n, z >=< x_n, T^*z >= 0$ and so $< y, z >= 0$. This proves that $\overline{R(T)} \subset N(T^*)^\perp$. To prove the opposite inclusion, we assume by contradiction that there exists $p \in N(T^*)^\perp$ but $p \notin \overline{R(T)}$. Then there is f in the Hilbert space X such that $< y, f >= 0$ for all $y \in \overline{R(T)}$ and $< p, f >= 1$. Let $x \in D(T)$ and *assume that $D(T)$ is dense in X*. Since $< x, T^*f >=< Tx, f >= 0$ for all $x \in D(T)$, $T^*f = 0$. Since $p \in N(T^*)^\perp$, $< p, y >= 0$ for all y with $T^*y = 0$. It follows that $< p, f >= 0$. This contradiction proves that $\overline{R(T)} \supset N(T^*)^\perp$.

Let X and Y be complex Hilbert spaces. An operator $U \in L(X, Y)$ is said to be unitary if $U^*U = I_X$ (the identity on X) and $UU^* = I_Y$ (the identity on Y). These two equations imply that $R(U) = Y$, $D(U^*) = Y$, and $R(U^*) = X$. Given $U \in L(X, Y)$, the following statements are equivalent: (1) U is unitary; (2) $R(U) = Y$ and U preserves the inner product; (3) U is an isometric mapping of X onto Y.

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6. Dedication

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